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GLOBAL DYNAMICS IN THE SIMPLEST MODELS OF THREE-DIMENSIONAL WATER-WAVE PATTERNS

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Abstract – We explore the idea that periodic or chaotic finite motions corresponding to attractors in the simplest models of resonant wave interactions might shed light on the problem of pattern formation. First we identify those dynamical regimes of interest which imply certain specific relations between physically observable variables, e.g. between amplitudes and phases of Fourier harmonics comprising the pattern. To be of relevance to reality, the regimes must be robust.

The issue of *structural stability of low-dimensional dynamical models* is central to our work. We show that the classical model of three-wave resonant interactions in a non-conservative medium is structurally unstable with respect to small cubic interactions. The structural instability is found to be due to the presence of certain extremely sensitive points in the unperturbed system attractors. The model describing the horse-shoe pattern formation due to non-conservative quintet interactions [11] is also analyzed and a rich family of attractors is mapped. The absence of such sensitive points in the found attractors thus indicates the robustness of the regimes of interest. Applicability of these models to the problem of 3-D water wave patterns is discussed. Our general conclusion is that extreme caution is necessary in applying the dynamical system approach, based upon low-dimensional models, to the problem of water wave pattern formation.

1. Introduction

Recent progress in mathematical modelling of pattern formation in various media is mainly based upon analysis of some low-dimensional models (see e.g. [3]). Commonly, such models are derived by means of an asymptotic procedure heavily exploiting smallness of nonlinearity and retaining the leading order nonlinear terms only. Other essential steps are *a priori* selection of a small number of effectively interacting normal modes and the assumption of spatial uniformity for continuous systems. The resulting low-dimensional system of ODEs constitutes a model which is the starting point for the dynamical system approach.

In the purely conservative case these models predict a mere periodic exchange of energy between different modes specified entirely by the initial conditions. Taking into account non-conservative effects enormously enriches the dynamics. The main new feature of non-conservative systems is the so-called *auto-oscillations,* broadly understood as periodic, multi-periodic and, sometimes, chaotic asymptotic regimes. In contrast to conservative systems, these regimes depend weakly or not at all on initial conditions. They correspond to some attractors in the system phase space. The cornerstone of the dynamical system approach is the mapping of such attractors and domains of their attraction as a function of the system parameters and thus obtaining an understanding of the large-time asymptotics of the system global dynamics. The existence of certain regular attractors is often interpreted as an indication of the existence of the patterns, while the specific relationships among the field variables typical for particular regimes prescribe the specifics of the patterns.

However, the relationship between the particular regimes of the sets of ODEs constituting a model and experimentally observed relevant patterns is not straightforward. The dynamical system approach aimed at studying *global dynamics* is commonly applied to a *given* dynamical system derived by means of an asymptotic expansion under certain assumptions, which are quite often valid for some particular regimes only. To be able to relate such a particular regular regime to the experimentally observed patterns one should ensure, *(i)* the robustness of the regime within the framework of the given dynamical system, and *(ii)* the structural stability of the system itself.

The issue of structural stability of the dynamical systems and particular regimes we address in the present work has not got the attention it merits. More specifically, we study global dynamics within the framework of two dynamical systems relevant to the problem of water wave pattern formation. For the regimes of interest we will try to find out

(i) Whether and when the higher order nonlinear terms, commonly neglected in the model derivation, could be of qualitative importance (the issue of model structural stability)?

(ii) Whether and when consideration could be confined to some particular regimes (to some sub-spaces in the phase space) despite the fact that the model proves to be inadequate for description of global dynamics?

The questions are very general indeed. However, in the present work we confine our consideration to the two simplest models which describe the formation of 3-D water wave patterns.

First we consider the most well-known of the non-trivial low-dimensional models, that of non-conservative three-wave resonant interaction (see e.g. [4,10,15]). It is relevant to all weakly-nonconservative media, where the dispersion law for the small-amplitude waves $\omega(\mathbf{K})$ permits resonant processes of the type

$$\begin{aligned} \mathbf{K}_i + \mathbf{K}_j &= \mathbf{K}_0 \\ \omega_r(\mathbf{K}_i) + \omega_r(\mathbf{K}_j) &\approx \omega_r(\mathbf{K}_0) \end{aligned} \tag{1}$$

where $\mathbf{K}_i$ are the wave-vectors of plane quasi-monochromatic waves, ω_r are real parts of the frequencies, and nonlinear and non-conservative effects (dissipation/amplification) are small, being of the same order of smallness ε as the wave amplitude. The complex amplitudes $a_i \equiv a(\mathbf{K})$ of the spatially uniform quasi-monochromatic waves involved into such triad interaction are governed by the classical model [4,10]

$$\begin{aligned} \mathrm{i}\dot{a}_0 &= U a_1 a_2 + \mathrm{i}\Gamma_0 a_o \\ \mathrm{i}\dot{a}_1 &= U a_0 a_2^* + \mathrm{i}\Gamma_1 a_1 \\ \mathrm{i}\dot{a}_2 &= U a_1^* a_0 + \mathrm{i}\Gamma_2 a_2 \end{aligned} \tag{2}$$

where U and Γ_i are the nonlinear interaction coefficient and dissipation ($\Gamma_i < 0$) / amplification ($\Gamma_i > 0$) rates, respectively. In the context of the water waves the amplification is due to wind, while negative Γ_i is due to the combined action of different types of dissipation (turbulence in the water and in the air and sink into smaller scales). At the leading order the variables a_i in Eq.(2) are either Fourier harmonics of primitive variables (surface elevation, velocity potential) or their linear combinations. The resonant conditions (1) can be satisfied for the water waves in the capillary and capillary-gravity ranges (see Fig.1*a*).

We show that perturbation of the classic three-wave resonant model (2), by taking into account small higher-order (cubic) self-interaction terms, leads to dramatic changes in the dynamics. This structural instability of the three-wave system is explained by the fact that some parts of the phase trajectories "controlling" transition between different regimes are very sensitive to small perturbations of the system.

The second model of 3-D patterns we focus upon was proposed in [11] to explain characteristic crescent water wave patterns (the so-called "horse-shoes"), first experimentally observed by Su *et al.* [12,13]. The "horse-shoe" model describes wave dynamics due to quintet resonant interactions of the type

$$\begin{aligned} \mathbf{K}_1 + \mathbf{K}_2 &= 3\mathbf{K}_0 \\ \omega_r(\mathbf{K}_1) + \omega_r(\mathbf{K}_2) &\approx 3\omega_r(\mathbf{K}_0) \end{aligned} \tag{3}$$

with input into the main wave $\mathbf{K}_0$ oriented alongwind and dissipation of two oblique satellites $\mathbf{K}_1, \mathbf{K}_2$ (see sketch in Fig.1b). The simplest version of such a model, when two satellites are assumed symmetric and equal,

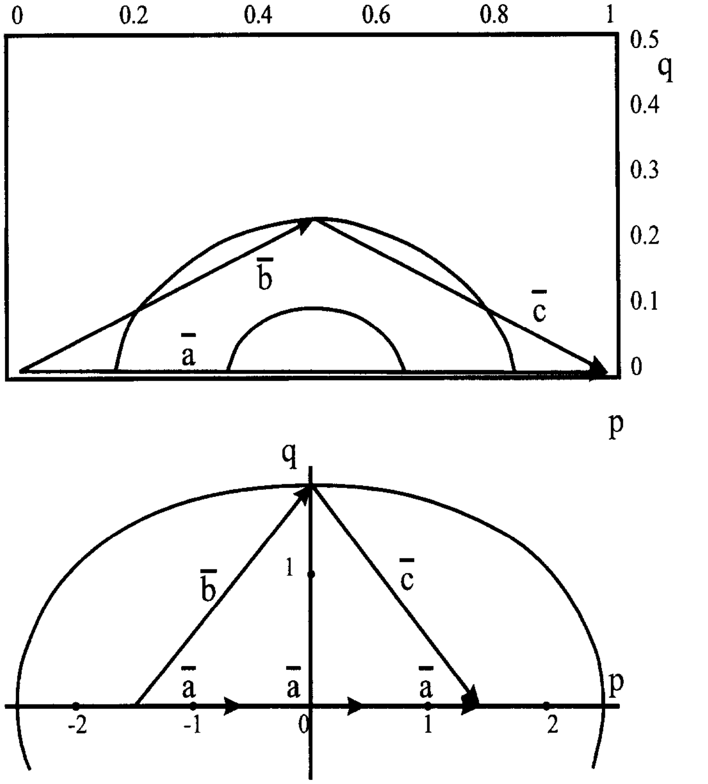


FIGURE 1. Geometric configuration of resonant interactions *(a)* — for three-wave resonant interaction of gravity-capillary waves for various Bond numbers B of the shortest wave **a**. For each B the oblique resonant components **b** and **c** lie on the corresponding oval curve; *(b)* — for five-wave resonant interaction of deep water gravity waves. The central downwind wave is **a**, the oblique satellites are **b** and **c**.

reads [11]

$$\begin{array}{rll} A_t = & 3WA^2B^2\sin\Phi & +\Gamma_A A \\ B_t = & -WBA^3\sin\Phi & +\Gamma_B B \\ \Phi_t = & \Delta + PA^2 + QB^2 & +W(9AB^2 - 2A^3)\cos\Phi \end{array} \tag{4}$$

Here A and B are moduli of the central and satellite harmonics respectively, while α and β are their phase angles and $\Phi = 3\alpha - 2\beta$. The expressions for the coefficients are given in [11]. The model, being much more complex and richer than the three-wave one, does not permit effective analytical study. Nevertheless, it proves to be possible to map the attractors and identify the regular regimes which could be responsible for the pattern formation. The regimes of interest prove to be free from the "sensitive points" thus indicating the system's structural stability.

The work is organized as follows. In § 2 we analyse the model of resonant gravity-capillary triad. We show that taking into account small self-interaction leads to qualitative changes in the system dynamics; in particular, it alters the sequence of the system bifurcations. In § 3 the "horse-shoe" model is analyzed and dynamical regimes of physical interest are identified. A sequence of bifurcations of the system is outlined as well. Discussion is centred on the applicability of the results for water wave pattern formation.

2. Three-wave model for gravity-capillary waves

Consider an extension of the classical three-wave resonant model (2) modified by taking into account small higher-order (cubic) self-interaction terms [15]. We cast it in terms of real amplitudes and phases as follows

$$\begin{aligned} \dot{A} &= \quad UBC\sin\Phi + \Gamma_A A; \\ \dot{B} &= -UAC\sin\Phi + \Gamma_B B; \\ \dot{C} &= -UAB\sin\Phi + \Gamma_C C; \\ \dot{\Phi} &= \delta + U(BC/A - AB/C - AC/B)\cos\Phi \quad + V_A A^2 + V_B B^2 + V_C C^2. \end{aligned} \tag{5}$$

where A, B, C are real amplitudes of wave components in (2), while the phases of the components enter the equations as a unique combination $\Phi = \phi_A - \phi_B - \phi_C$. The parameter $\delta = \omega_0 - \omega_1 - \omega_2$ is a linear frequency mismatch. Three-wave resonance coefficient $U = U^{(1)}(\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2)$ and self-interaction coefficients V_A, V_B, V_C can be found in [9] for the case of gravity-capillary waves. We assume $\Gamma_A > 0$ (generation) and $\Gamma_B, \Gamma_C < 0$. The additional terms in the system (5) as compared to the classical model (2) are due to the self-interaction and are $O(\varepsilon)$ small in comparison to all the others.

In the absence of generation/dissipation the system (5) allows for three conservation laws and is integrable. The solutions in terms of elliptic functions describe periodic exchange of energy between three components of the system (e.g. [4]) but will play no role in our further analysis.

We consider only the symmetric case, i.e. assume $B = C$ and $\Gamma_B = \Gamma_C$. It is commonly assumed [4,15] that the full system (5) tends quite rapidly to the symmetric state $B = C$ and, thus, we can confine our consideration to the system of just three ODEs.

Introducing non-dimensional variables based on the decay rate Γ_B

$$\tau = |\Gamma_B| t; \qquad \tilde{A} = A|\Gamma_B|/U; \qquad \tilde{B} = B|\Gamma_B|/U. \tag{6}$$

(τ is "new" time) and the transformation [14]

$$X = \tilde{A}\cos\Phi; \qquad Y = \tilde{A}\sin\Phi; \qquad Z = \tilde{B}^2. \tag{7}$$

we arrive at the following dynamical system

$$\begin{aligned} \dot{X} &= \gamma X - \delta Y + 2XY \quad -PY(X^2+Y^2) - QYZ \\ \dot{Y} &= \quad \delta X + \gamma Y + Z \quad -2X^2 + PX(X^2+Y^2) + QXZ \\ \dot{Z} &= \quad -2Z - 2YZ. \end{aligned} \tag{8}$$

The main advantage of transformation (7) to variables X, Y, Z lies in the fact that it leads to ODEs with polynomial right-hand sides which are more convenient for analysis. Note also the relation of variables X, Y, Z in Eq.(8) to the variables $a_i(\mathbf{k})$ in (2): X and Y are just real and imaginary parts respectively of the main harmonic amplitude a_0, while Z is the second harmonic modulo squared. In these variables shrinking of the phase volume, which characterises the degree of "dissipativeness" of the system, is especially easily to find. Indeed, one straightforwardly obtains from (8)

$$\mathrm{div}R\vec{H}S \equiv \mathrm{div}\dot{X} = \frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = 2(\gamma - 1) \tag{9}$$

The nonlinear terms are conservative and drop out. Thus, at $\gamma \sim 1$, when generation is balanced by dissipation, the system (8) is close to conservative ones. For $\gamma < 1$ the phase volume shrinks and attractive subspaces can exist in the phase space.

The case $P = 0, Q = 0$ corresponds to the classical model (2) studied in detail both analytically and numerically [4, 14, 16].

In the extended model (8) the cubic coefficients P and Q are additional parameters and the system dynamics does depend on their values. However we, in line with the previous studies (e.g. [16]), focus on the dependence on the single parameter $\gamma = \Gamma_A/\Gamma_B$ considering other parameters P, Q and δ fixed. The particular importance of γ can be seen from (9) that shows that the phase volume evolution is specified entirely by the single parameter γ. Our goal is to show a drastic effect of small but non-zero additional parameters P, Q on the system dynamics and, in particular, on the bifurcation sequence of the system dynamical regimes in terms of γ, while a detailed study of the influence of the cubic terms goes beyond the framework of the present work.

Vyshkind & Rabinovich [14] have found a way to advance analytically in constructing solutions to the three-wave resonance model utilizing the two-scale temporal structure of these solutions. This approach is based on a very simple and attractive idea, that phase volume shrinking can be strongly anisotropic in the dissipative system. The system can exhibit its "conservative features" in one direction, while its "non-conservative nature" can prevail in an orthogonal subspace. This idea was further considerably developed in [5–7].

2.1. *Scaling. Two-scale evolution*

The most interesting dynamical regimes correspond to relatively small values of γ [5, 16]. This fact was used in [5, 14] to construct "conservative" and "non-conservative" subspaces for the model (2). Assuming $\gamma \ll 1$, we first scale solutions as follows [14]

$$X \sim O(1); \quad Y \sim O(1); \quad Z \sim O(\gamma) \tag{10}$$

This means that X, Y evolve in "fast-time" regime while small Z varies slowly and its variation can be taking into account by averaging "fast-time" dependence on X and Y. The corresponding "fast-time" equations for X, Y possess conservation law. One obtains a simple rotation in the (X, Y) plane. Nonlinearity of the governing equations (5) is responsible for this rotation only. The amplitudes of both harmonics A and B are not changing in this approximation and the phase angle Φ evolves gradually. The "fast-time" approach remains valid as long as the right-hand sides of the equations for X and Y in (8) are not small. For the case of small self-interaction and small γ the RHS vanishes near the plane

$$X_* = \delta/2$$

where transition to "slow" regime of motion occurs [14]. The "slow" evolution of the system is confined to the vicinity of the plane $X \approx X_*$ in phase space. This fact greatly simplifies dynamics and allows one to describe it analytically. Amplitudes A and B and phase angle Φ vary dramatically in this regime and these variations are due to joint effect of dissipation and nonlinearity.

Consider first, in more detail, the classical three-wave system, i.e. with zero self-interaction terms P and Q. It is easy to see that the system has just two stationary points. The trivial stationary state $X_0 = Y_0 = Z_0 = 0$ is a saddle-focus. The corresponding eigenvalues are

$$\lambda_1 = -2; \qquad \lambda_{2,3} = \gamma \pm i\delta. \tag{11}$$

The second stationary point is located at

$$X_1 = \delta/(2-\gamma); \qquad Y_1 = -1; \qquad Z_1 = \gamma[1 + \delta^2/(2-\gamma)^2]. \tag{12}$$

and its eigenvalues in the limit $\delta \to 0$ take the form

$$\lambda_1 = \gamma - 2; \qquad \lambda_{2,3} = \frac{1}{2}\left(\gamma \pm \sqrt{\gamma^2 + 8\gamma}\right) \tag{13}$$

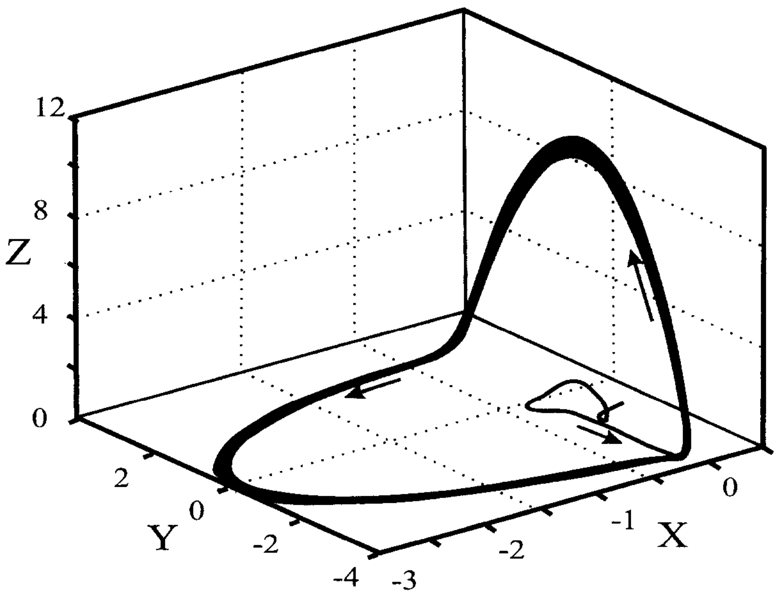


FIGURE 2. Phase trajectory for the classical model of three-wave resonance illustrating an attractor corresponding to a limit cycle. Parameters: $\gamma = 0.25, \delta = -0.2$.

One can see from (11,13) that as γ diminishes, the stationary points become less unstable and, this can explain qualitatively the series of the system bifurcations to multi-periodic cycles and, eventually, to "strange" attractor.

The above tentative qualitative consideration is justified by direct simulations. Fig. 2 illustrates a solution to (8) with $P = Q = 0$ shown in the X, Y, Z phase space. Its parameters correspond to those given in [16] for two-periodic limit cycle ($\gamma = 0.1$, $\delta = -0.2$). Initial conditions are taken in the proximity of non-trivial stationary state (12), but the form of the attractor does not depend on initial conditions. Periodic motion corresponding to the attractor is confined to the half-space $X < X_* = \delta/2$. This is of prime importance in the context of wave pattern formation: the phase difference between the high-frequency harmonic A and low-frequency one B always remains in a specific angle sector, which implies the wave fronts of corresponding oscillating wave patterns are crescent-shaped and always oriented forward.

Taking into account small self-interaction parameters P, Q does not affect the fact of existence of attractors but can, as we demonstrate below, change dramatically their forms.

2.2. *Example of bifurcation sequence for three-wave model with self-interaction*

Consider again the system (8) with nonzero $O(\varepsilon)$ self-interaction terms, $P \neq 0$ and $Q \neq 0$. The system preserves the same trivial stationary state $X_0 = Y_0 = Z_0 = 0$ and a slightly shifted non-trivial fixed point (12). The new fixed point (X_2, Y_2, Z_2) comes from infinity

$$X_2 \approx (2-\gamma)/(P+\gamma Q); \qquad Y_2 \approx= -1; \qquad Z_2 \approx \gamma(\gamma-2)^2/(P+\gamma Q)^2. \tag{14}$$

Explicit expressions for the instability exponents at this point can be easily obtained for the case $\delta = 0$, $P = 0$. Since our simulations show that this case contains all qualitative effects of the complete model (8) we confine ourselves mainly to its consideration. The type of the new fixed point is specified by solutions of the characteristic equation

$$\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0 \tag{15}$$

with coefficients

$$c_0 = 2(\gamma-2)^3/(\gamma Q^2); \qquad c_1 = (\gamma-2)^2(\gamma^2+2\gamma+2)/(\gamma^2 Q^2); \qquad c_2 = 2(1-\gamma).$$

Approximate solutions to Eq.(15) can be easily found

$$\begin{aligned} \lambda_1 &= 2\gamma(2-\gamma)/(\gamma^2+2\gamma+4); \\ \lambda_{2,3} &= (\gamma^3+2\gamma^2-4)/(\gamma^2+2\gamma+4) \pm \quad i(\gamma-2)\sqrt{\gamma^2+2\gamma+4}/(Q\gamma). \end{aligned} \tag{16}$$

As it is seen from Eq.(16) for $\gamma < 1$ the stationary point is a saddle-focus. In the unstable direction the instability rate λ_1 is of order of unity, while the attraction rate is given by real part of $\lambda_{2,3}$. Near this point trajectories are turning around very rapidly ($\mathrm{Im}(\lambda_{2,3}) \sim Q^{-1}$). *A priori*, one may expect the effect of such a stationary point to be negligible since the point comes from infinity and is well separated from the regimes of interest, the distance to the reference frame origin being of order Q^{-2} while exponents λ_i are of order of unity. However, our simulations show that in the presence of this new stationary point the system dynamics changes qualitatively. This should not be misinterpreted as a direct influence of the new fixed point. It is better to say that the effect of cubic nonlinearities results at the same time in emerging of the new fixed point *and* small deformation of the phase space structure in the domain where the bounded regimes under consideration occur. We stress that the cubic nonlinearities always remain small for these regimes.

As an example we present here a sequence of the bifurcations of the system. For the calculations we fix the parameters $\delta = -0.2$, $Q = 0.01$, $P = 0$ and change generation/dissipation rates γ only, to compare with the established results for the unperturbed case. At $\gamma = 0.25$ we have again one-periodic limit cycle (see Fig. 3), but essentially distinct from one for $Q = P = 0$ shown in Fig. 2. The "fast" motion is now confined to $X > \delta/2$ that is to the opposite half-space compared to its counterpart with the same parameters in the classical model. This implies, in particular, that the corresponding gravity-capillary "horse-shoe" patterns will be oriented *backward* most of the time, in contrast to the forward orientation predicted by the classical model. This qualitative change of attractors occurs when the cubic terms are very small; indeed the threshold value of the coefficient was found to be $Q = 2.65 \cdot 10^{-4}$.

This extraordinary sensitivity of the system effectively means structural instability of the classical three-wave model, although not precisely in the classical sense: there is a *very small but finite threshold.* This structural instability is due to the fact that the system has exponentially diverging segments of trajectories and is accumulating perturbations near the points of transition from the "fast" to "slow" regimes. Since the attractors correspond to large-time system asymptotics such an accumulation of perturbations may result in dramatic consequences. It should be noted also that in contrast to the attractors of the classic system, which are very sensitive to noise, the new attractors appear in our simulations to be much more robust.

Generally speaking, demonstration of just one example, of the type exhibited by Figs. 2, 3 is already sufficient to claim structural instability of the classical three-wave model. This makes the results of all earlier mappings of the regimes within its framework irrelevant and raises the question about the true sequence of the regimes. Below we briefly outline a few interesting points in a "true" bifurcation sequence derived within the framework of the model with self-interaction, the details of which will be reported elsewhere.

Two-periodic limit cycle appears at $\gamma = 0.1525$, it corresponds to temporally alternating (forward – backward) wave pattern geometry. One part of the cycle resembles the case of Fig. 2 with $X < X_*$, another part corresponds to "backward" patterns with $X > X_*$. After a series of bifurcations to multi-periodic and chaotic ("strange attractor") regimes a robust one-periodic attractor (limit cycle) appears at $\gamma = 0.125$ corresponding to forward oriented wave patterns (see Fig. 2).

At $\gamma = 0.075$ we encounter a particular case when two different limit cycles, corresponding to the opposite orientation of wave patterns, emerge depending on the initial conditions (Fig. 4). Further reducing the parameter γ leads to a new series of bifurcations for limit cycles with backward orientation of wave patterns.

The presented examples do not only show clearly structural instability of the classical three-wave model with respect to unavoidable small self-interaction terms, but also illustrate the richness of "true" attractors

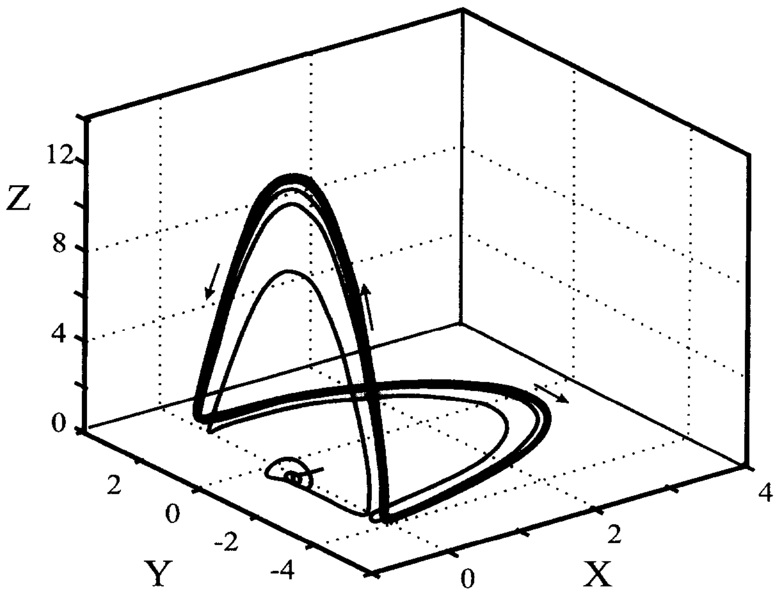


FIGURE 3. Phase trajectory illustrating limit cycle in the model of three-wave resonance with self-interaction. Parameters are the same as in Fig.2 $\gamma = 0.25, \delta = -0.2, Q = 0.01$. Stress that the one-periodic limit cycle orientation is opposite to that of the previous figure.

in a bifurcation sequence and thus potential richness of the corresponding family of 3-D wave patterns. Our conclusion on the system structural instability is in line with the results of [5], where extreme sensitivity of the classical three-wave model with respect to external noise was found for small γ and attributed to the presence of exponentially diverging segments of trajectories in the system attractor. In fact, the effect of cubic terms may be viewed as that of a sign-definite persistent perturbation. Not suprisingly, because of the sign-definitness and persistence the influence of cubic terms is more pronounced and is not confined to the range of small γ.

3. The "horse-shoe"pattern model

The analysis carried out above for the three-wave model with self-interaction was facilitated essentially by the fact that the type of "new" remote stationary point of the system was always the same (saddle-focus) in the whole parameter domain. For the model (4) we consider in this section the situation is much more complicated — both the number of stationary points and their types depend on the system parameters, moreover, the number of "control" parameters increases.

In the spirit of the previous section, upon rescaling variables

$$\tau = |\Gamma_B| t; \qquad \tilde{A} = (\Gamma_B/W)^{1/3} A; \qquad \tilde{B} = (\Gamma_B/W)^{1/3} B. \tag{17}$$

and performing the transformation (6), one arrives at the following system with polynomial right-hand sides

$$\begin{aligned} \dot{X} &= \gamma X - \delta Y - 6XYZ + 2XY(X^2+Y^2) - \varepsilon^{-1}\left[Y(X^2+Y^2) + RYZ\right]; \\ \dot{Y} &= \delta X + \gamma Y + 3Z(Y^2+3X^2) - 2X^2(X^2+Y^2) + \varepsilon^{-1}\left[X(X^2+Y^2) + RXZ\right]; \\ \dot{Z} &= 2Z - 2YZ(X^2+Y^2). \end{aligned} \tag{18}$$

Non-dimensional parameters of self-interaction are $1/\varepsilon = -P/(W^2\Gamma_B)^{1/3}$ and $R = Q/P \approx 4.69/(W^2\Gamma_B)^{1/3}$. Here ε is a rescaled parameter still proportional to wave amplitude at fixed dissipation rate, while coefficients $P = 0.0812$ and $Q = 0.0376$ are universal constants for deep-water waves under this scaling.

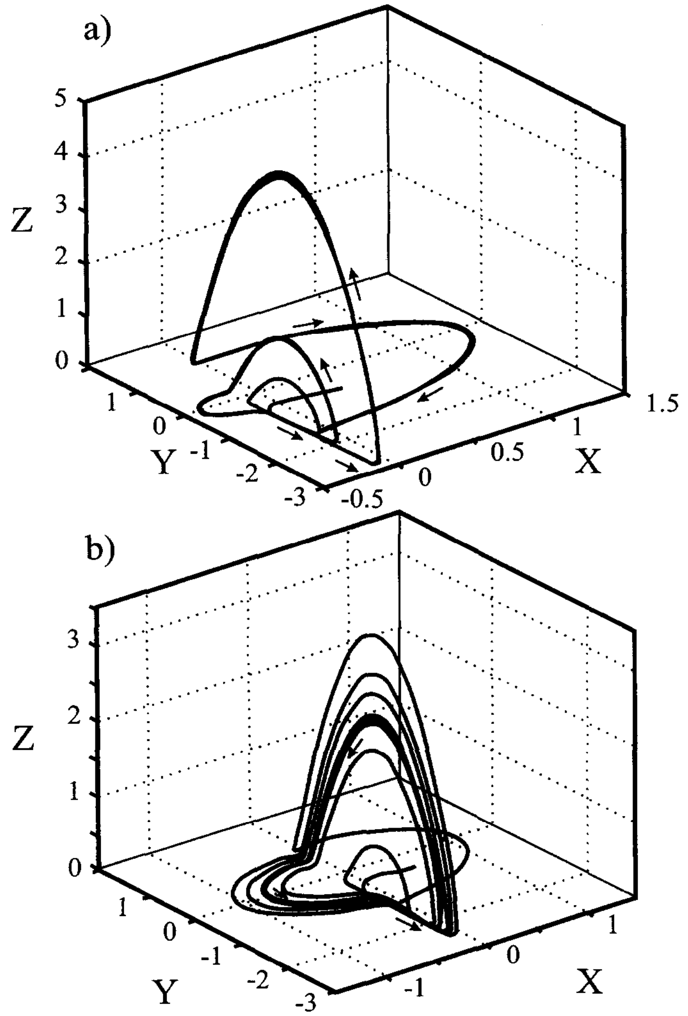


FIGURE 4. Coexistence of two different attractors in the three-wave model with self-interaction ($\gamma = 0.075, \delta = -0.2, Q = 0.01$). *(a)* — Attractor of the "classical three-wave" type limit cycle emerges out of initial conditions $X = 0.25, Y = -1, Z = 0.29$. *(b)* — Initial conditions $X = 0.25, Y = -1, Z = 0.31$ evolve into attractor typical of the model with self-interaction.

Our first step is to determine properties of the stationary points as a function of the system parameters and then proceed with analysis of the system dynamics.

The trivial stationary point, $X_0 = Y_0 = Z_0 = 0$, is a saddle-focus as in the previous cases (see Eq.11). For non-trivial stationary states it can easily be found

$$X_N = \frac{\delta Y_N - \varepsilon^{-1}[1 + Q\gamma/(3P)]}{3\gamma - 2}; \qquad Z_N = -\gamma/(3Y_N). \tag{19}$$

Here Y_* obeys the cubic equation

$$Y_N^3 + \left(\frac{\delta Y_N - \varepsilon^{-1}[1 + Q\gamma/(3P)]}{3\gamma - 2}\right)^2 Y_N + 1 = 0 \tag{20}$$

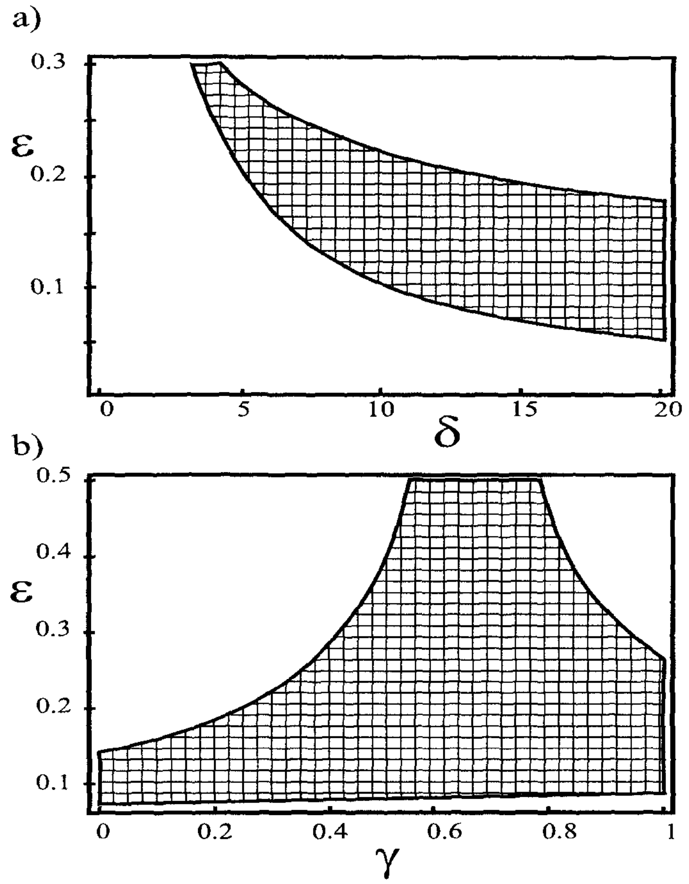


FIGURE 5. Stationary points in the horse-shoe model. Domains correponding to three stationary points are shaded. Two cross-sections of the 3-D parameter space are given. *(a)* At fixed $\delta = 14$. *(b)* At fixed $\gamma = 0.25$.

The three-dimensional space of parameters δ, γ and ε can be subdivided into domains depending on the number of real roots of Eq.(20). Cross-sections of these domains for fixed values of γ and δ are presented in Fig. 5. In unshaded domains of parameters there is only one stationary state. When this state is stable the wave dynamics of the corresponding wave pattern is quite simple — the system eventually reaches this stationary state from rather wide domain of initial conditions. In our study this case, as a rule, corresponds to very high wave amplitudes ($ak > 0.3$) and, thus, is of limited physical interest.

In the shaded domain of parameters shown in Fig. 5 there are three fixed points in addition to the trivial equilibrium $X_0 = Y_0 = Z_0 = 0$. All possible combinations of these points of different types are of importance for our study. Generally, there always exists one stable point, corresponding to very high wave amplitudes. When there is also an additional stable fixed point (Fig. 6), the phase space is divided into two domains of attraction and the final state of the system depends essentially on its initial conditions. The stationary point A shown in Fig. 6 corresponds to steepness $(ak) \approx 0.25$. Another stable stationary point has unrealistically high wave steepness ≈ 2.5 and is not shown in the figure. Note that these two stationary points are always in opposite quadrants of the plane (X, Y). This implies quite different phase relations between the central harmonic and the satellites. Thus, within the framework of the model the final state of the system can depend strongly on the initial phase relations. Depending on the initial conditions the system can evolve either into a stationary state A, adequately described by the model, or to states lying beyond the applicability of the model.

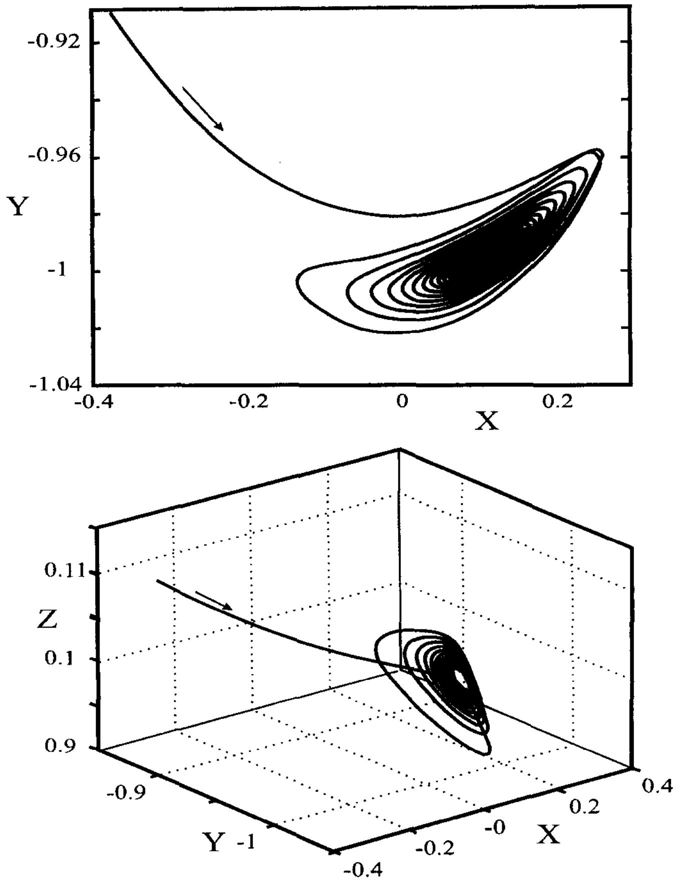


FIGURE 6. An example of phase trajectory in the horse-shoe model attracted by a stationary point A located at $\gamma = 0.3, \delta = 11.8, \varepsilon = 0.09$. The case corresponds to wave steepness $(ak) \approx 0.25$.

At a certain range of parameters the stable stationary point A shown in Fig. 6 loses its stability and a limit cycle, corresponding to stable periodic motion, becomes the only attractor as is shown in Fig. 7. Some important features of these dynamical regimes are worth stressing:

- They appear at smaller wave amplitudes as compared to stable stationary states considered above and have larger domains of attraction;
- The limit motions occur in a relatively narrow range of amplitudes and angles Φ.

Further bifurcations lead to stable multi-periodic limit cycles (Fig. 8).

Typical of the presented examples is a strong dependence of the dynamics on the initial states of the system. Depending on the initial conditions the system can evolve to a state beyond the model validity, thus indicating necessity of the improvement by means of more realistic description of the non-conservative processes. In fact, the effect of nonlinear dissipation owing to wave breaking should be incorporated into the model to extend its range of validity, which is unlikely to be done in a remotely rigorous way in the foreseeable future. Despite the mentioned shortcomings of the model *some* of the regimes described adequately within the framework of the model do have chances to be related to those observed in numerical simulations and experimental studies of 3-D water wave patterns.

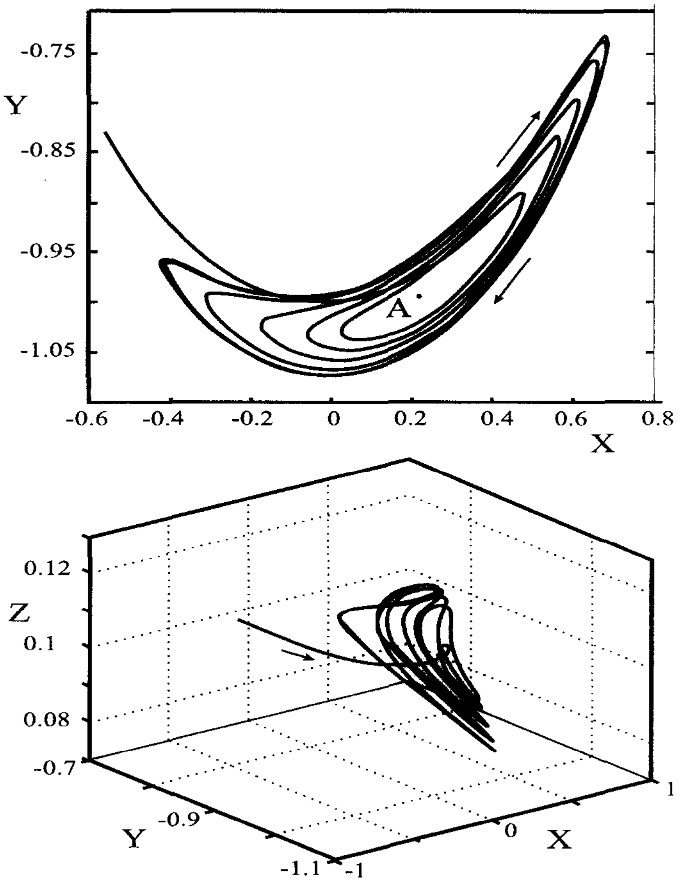


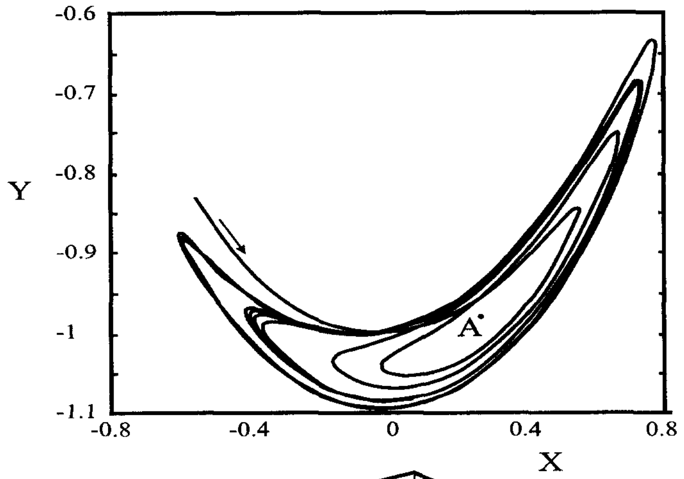
FIGURE 7. An example of phase trajectory in the horse-shoe model approaching one-periodic limit cycle at $\gamma = 0.3, \delta = 13.6, \varepsilon = 0.07$. The case corresponds to wave steepness $(ak) \approx 0.2$.

4. Discussion

In the present study we demonstrated immensely rich dynamics in the simplest models of wave resonant interaction. Depending on the relatively small number of parameters the dynamics can be periodic or aperiodic in a finite domain of the system phase space. Can the found regimes describe the real 3-D water wave patterns?

For the situations described by the three-wave model our answer is — very unlikely. The simplest toy-model was used to demonstrate danger of common dynamical system approach which starts with a given model. The classic three-wave model is quite correctly derived asymptotically to be valid at time spans $O(\varepsilon^{-1})$. However to describe its asymptotic behaviour at large times, it proves to be necessary to take into account small self-interaction terms as if we were attempting to describe $O(\varepsilon^{-2})$ evolution. The main gap between the results of the three-wave models and water wave reality lies in the very fact which makes them so attractive for analysis, the low number of modes. Except for very special situations created intentionally in laboratory experiments, the number of interacting modes does not remain confined to just a few.

The perspectives of the applicability of the "horse-shoe" model results look brighter, despite the proven structural instability of the model with respect to inclusion of extra modes [1] and the discussed above shortcomings in the description of generation/dissipation. Some of the found regimes might prove robust even within the framework of full hydrodynamic equations, since the range of amplitude variations specific to the particular



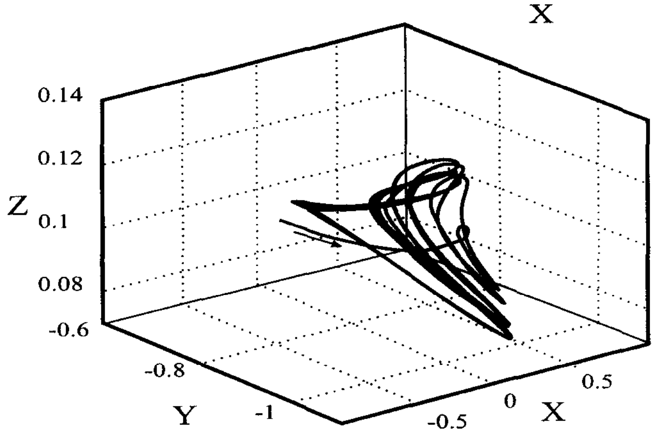


FIGURE 8. An example of two-periodic limit cycle in the horse-shoe model: $\gamma = 0.3, \delta = 13.85, \varepsilon = 0.07$. The case corresponds to wave steepness $(ak) \approx 0.2$.

regime may ensure essentially the same low-modal character of field evolution due to the selection mechanism found in [1]. No simulation test of this kind has been carried out so far.

In brief, our main conclusion is that extreme caution is recommended in applying predictions of the low-dimensional models to the problem of 3-D water wave pattern formation.

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